Homework 6

Name: **SOLUTIONS** Date: July 22, 2015

P 1. Find the value of the following improper integral. If it diverges, state so and explain why.

$$\int_0^8 \frac{3}{\sqrt{8-x}}\,dx$$

Solution: Note,

$$\begin{aligned}\int \frac{3}{\sqrt{8-x}}\,dx &= \int \frac{3}{\sqrt{u}}\,\frac{du}{-1}\\ &= -3\int u^{-1/2}\,du\\ &= -6\sqrt{u}+C\\ &= -6\sqrt{8-x}+C\end{aligned}$$

and

$$\begin{aligned}\int_0^b \frac{3}{\sqrt{8-x}}\,dx &= -6\sqrt{8-x}\Big|_0^b\\ &= -6\sqrt{8-b}+12\sqrt{2}\end{aligned}$$

So,

$$\begin{aligned}\int_0^8 \frac{3}{\sqrt{8-x}}\,dx &= \lim_{b\to 8^-}\int_0^b \frac{3}{\sqrt{8-x}}\,dx\\ &= \lim_{b\to 8^-}\left(-6\sqrt{8-b}+12\sqrt{2}\right)\\ &= 12\sqrt{2}-6\lim_{b\to 8^-}\sqrt{8-b}\\ &= 12\sqrt{2}\end{aligned}$$

P 2. Find the limit.

$$\lim_{x\to a^+}\left(1+\frac{1}{x}\right)^x$$

where

$$\begin{aligned}
a &= \lim_{x\to b}\frac{(3x+2)^{200}(2x-1)}{(x+1)^{202}}\\
b &= \lim_{x\to c^+}\frac{(x^2+9x)\sin x}{(x-23)(x+1)}\\
c &= \lim_{x\to 0^+}\frac{\sin 23x}{x}
\end{aligned}$$

Solution:

$$\begin{aligned}
c &= \lim_{x\to 0^+}\frac{\sin 23x}{x}\\
&\overset{0/0}{=} \lim_{x\to 0^+}\frac{23\cos x}{1}\\
&= 23
\end{aligned}$$

$$\begin{aligned}
b &= \lim_{x\to c^+}\frac{(x^2+9x)\sin x}{(x-23)(x+1)}\\
&= \lim_{x\to 23^+}\frac{(x^2+9x)\sin x}{(x-23)(x+1)}\\
&= \lim_{x\to 23^+}\frac{(x^2+9x)\sin x}{x+1}\cdot\lim_{x\to 23^+}\frac{1}{x-23}\\
&= \underbrace{\frac{92\sin(23)}{3}}_{\text{negative}}\cdot\underbrace{\lim_{x\to 23^+}\frac{1}{x-23}}_{\infty}\\
&= -\infty
\end{aligned}$$

$$\begin{aligned}
a &= \lim_{x\to b}\frac{(3x+2)^{200}(2x-1)}{(x+1)^{202}}\\
&= \lim_{x\to -\infty}\frac{(3x+2)^{200}(2x-1)}{(x+1)^{202}}\\
&= \lim_{x\to -\infty}\frac{[x(3+2/x)]^{200}(2x-1)}{[x(1+1/x)]^{202}}\\
&= \lim_{x\to -\infty}\frac{[3x]^{200}(2x-1)}{[x]^{202}}\\
&= \lim_{x\to -\infty}\frac{3^{200}\cdot x^{200}(2x-1)}{x^{202}}\\
&= 3^{200}\lim_{x\to -\infty}\frac{2x-1}{x^2} = 3^{200}\lim_{x\to -\infty}\frac{2-1/x}{x}\\
&= 3^{200}\lim_{x\to -\infty}\frac{2}{x} = 3^{200}\cdot 0 = 0
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to a^+}\left(1+\frac{1}{x}\right)^x &= \lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x \\
&\overset{\infty^0}{=} \lim_{x\to 0^+} e^{\ln\left(\left(1+\frac{1}{x}\right)^x\right)} \\
&= e^{\lim_{x\to 0^+}\ln\left(\left(1+\frac{1}{x}\right)^x\right)}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0^+}\ln\left(\left(1+\frac{1}{x}\right)^x\right) &= \lim_{x\to 0^+} x\ln\left(1+\frac{1}{x}\right) \\
&\overset{0\cdot\infty}{=} \lim_{x\to 0^+}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&\overset{\infty/\infty}{=} \lim_{x\to 0^+}\frac{\frac{1}{1+\frac{1}{x}}(-1/x^2)}{-1/x^2} \\
&= \lim_{x\to 0^+}\frac{1}{1+\frac{1}{x}} = 0
\end{aligned}$$

So,

$$\lim_{x\to a^+}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to 0^+}\ln\left(\left(1+\frac{1}{x}\right)^x\right)} = e^0 = 1$$