Homework 5

Name: **SOLUTIONS** Date: July 22, 2015

P 1. Find the indefinite integral.

$$\int \frac{1}{16x^2+9}\,dx$$

Solution: Let $x = \frac{3}{4}\tan t$. Then $dx = \frac{3}{4}\sec^2 t\,dt$. Note that,

$$\frac{1}{16x^2+9} = \frac{1}{16(3/4\tan t)^2+9} = \frac{1}{9\sec^2 t} = \frac{1}{9}\cos^2 t \tag{1}$$

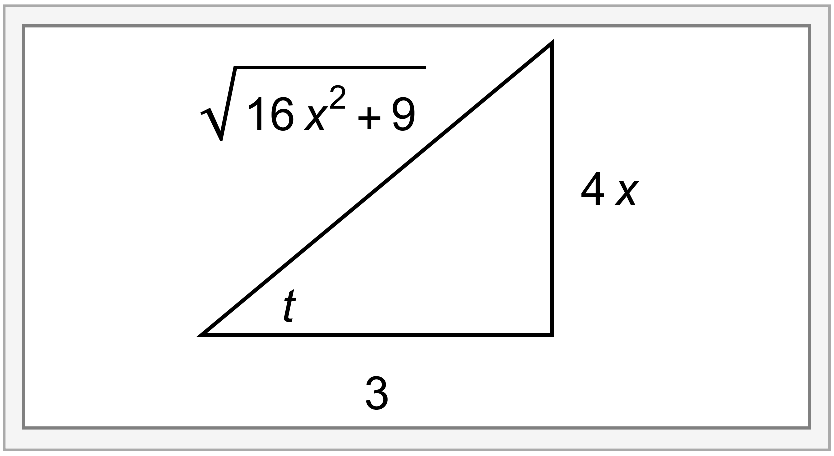
Using (1), we obtain

$$\int \frac{1}{16x^2+9}\,dx = \int \frac{1}{9}\cos^2 t\,\frac{3}{4}\sec^2 t\,dt = \frac{1}{12}\int\,dt = \frac{1}{12}t + C \tag{2}$$

Since $x = 3/4\tan t$ then

$$t = \arctan\left(\frac{4x}{3}\right) \tag{3}$$

If we needed any of the six basic trigonometric functions of t, we can use the triangle below, constructed from our original substitution. For example, $\sin t = \frac{4x}{\sqrt{16x^2+9}}$.



$$\int \frac{1}{16x^2+9}\,dx = \frac{1}{12}t + C = \frac{1}{12}\arctan\left(\frac{4x}{3}\right) + C$$

P 2. Find the indefinite integral.

$$\int \frac{x^5+2x^4+2x^3+2x^2-2x-45}{x^4+2x^3+x^2+8x-12}\,dx$$

Solution: By long division,

$$\frac{x^5+2x^4+2x^3+2x^2-2x-45}{x^4+2x^3+x^2+8x-12} = x + \frac{x^3-6x^2+10x-45}{x^4+2x^3+x^2+8x-12} \tag{4}$$

By inspection, we find that $x=1$ and $x=-3$ are zeros of the denominator of the fraction on the right-hand side of equation (4). By long or synthetic division, we obtain

$$x^4+2x^3+x^2+8x-12 = (x-1)(x+3)\left(x^2+4\right) \tag{5}$$

So, for some constants A, B, C and D,

$$\frac{x^3-6x^2+10x-45}{x^4+2x^3+x^2+8x-12} = \frac{x^3-6x^2+10x-45}{(x-1)(x+3)\left(x^2+4\right)} = \frac{A}{x-1}+\frac{B}{x+3}+\frac{Cx+D}{x^2+4} \tag{6}$$

By clearing the denominator in (6), we obtain

$$x^3-6x^2+10x-45 = A(x-3)(x^2+4)+B(x-1)(x^2+4)+(Cx+D)(x-1)(x+3) \tag{7}$$

Evaluating (7) at $x=1$, we obtain $A=-2$. Evaluating (7) at $x=-3$, we obtain $B=3$. Substituting these values into (6) we obtain

$$x^3-6x^2+10x-45 = (C+1)x^3+(2C+D-9)x^2+(-3C+2D+4)x-3(D+12) \tag{8}$$

So, $1=C+1$, $-6=2C+D-9$, $10=-3C+2D+4$, and $-45=-3(D+12)$. The first equation tells us that $C=0$ and the last equation tells us that $D=3$.

By (4) and (6) we obtain

$$\begin{aligned}
\int x+\frac{x^3-6x^2+10x-45}{x^4+2x^3+x^2+8x-12}\,dx &= \int x+\frac{-2}{x-1}+\frac{3}{x+3}+\frac{3}{x^2+4}\,dx \\
&= \frac{x^2}{2}-2\ln|x-1|+3\ln|x+3|+3\int\frac{1}{x^2+4}\,dx \\
&= \frac{x^2}{2}+\ln\left|\frac{(x+3)^3}{(x-1)^2}\right|+\frac{3}{4}\int\frac{1}{(x/2)^2+1}\,dx \\
&= \frac{x^2}{2}+\ln\left|\frac{(x+3)^3}{(x-1)^2}\right|+\frac{3}{4}\int\frac{1}{u^2+1}\,2du \\
&= \frac{x^2}{2}+\ln\left|\frac{(x+3)^3}{(x-1)^2}\right|+\frac{3}{2}\arctan u+C \\
&= \frac{x^2}{2}+\ln\left|\frac{(x+3)^3}{(x-1)^2}\right|+\frac{3}{2}\arctan(x/2)+C
\end{aligned} \tag{9}$$