Homework 2

Name: **SOLUTIONS** Date: May 31, 2015

P 1. Let c be greater than 4 and an x-coordinate of the points of intersection of the two curves given by $f(x) = x^3 + 72$ and $g(x) = 11x^2 - 10x$. Is

$$F(x) = \begin{cases} \sqrt{x} + 3, & x \le c \\ |x-2| - 1, & x > c \end{cases}$$

continuous at c? Explain.

Solution: To find the x-coordinates of the points of intersection of the two curves, set them equal to one another and solve.

$$\begin{aligned} x^3 + 72 &= 11x^2 - 10x \\ x^3 + 72 - 11x^2 + 10x &= 0 \\ p(x) = x^3 - 11x^2 + 10x + 72 &= 0 \end{aligned}$$

Notice that $p(x)$ is continuous on the closed interval $[8, 10]$ and that $p(8) = -40$ while $p(10) = 72$. By the Intermediate Value Theorem, $p(x)$ must have a zero in this interval. If the zero is rational, then it should be among the list of rational zeros of $p(x)$ which are:

$$x = \pm 1, \pm \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12, \pm 18, \pm 24, \pm 36, \pm 72$$

We see that $x = 9$ is a zero of $p(x)$ by checking that $p(9) = 0$. So $x - 9$ is a factor of $p(x)$. By synthetic or long division, we find that

$$x^3 - 11x^2 + 10x + 72 = (x-9)(x^2 - 2x - 8) = (x-9)(x-4)(x+2) = 0.$$

So, $c = 9$.
For

$$F(x) = \begin{cases} \sqrt{x} + 3, & x \le 9 \\ |x-2| - 1, & x > 9 \end{cases}$$

to be continuous at 9,

(a) $F(9)$ must be defined. We have that $F(9) = 6$.

(b) $\lim_{x \to 9} F(x)$ must exist and be finite. We have that $\lim_{x \to 9} F(x) = 6$

   (i) $\lim_{x \to 9^+} F(x) = \lim_{x \to 9^+} (|x-2| - 1) = \lim_{x \to 9^+} (x-3) = 6$

   (ii) $\lim_{x \to 9^-} F(x) = \lim_{x \to 9^-} (\sqrt{x} + 3) = 6$

(c) $\lim_{x \to 9} F(x) = F(9)$. By items (a) and (b), we have that this equality holds.

P 2. Find constants a and b such that

$$f(x) = \begin{cases} ax^2 - 3, & x \leq -3 \\ \sin(\pi x) + b, & x > -3 \end{cases}$$

is continuous everywhere.

Solution: The only place $f(x)$ can be discontinuous is at $x = -3$, since for $x < -3$ the function is a polynomial and for $x > -3$ the function is a sine function.

In order for f to be continuous at -3,

(a) $f(-3)$ must be defined. We have that $f(-3) = 9a - 3$.

(b) $\lim_{x \to -3} f(x)$ must exist and be finite. We have that $\lim_{x \to -3} f(x) = 9a - 3$

(i) $\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (\sin(\pi x) + b) = \sin(-3\pi) + b = b$

(ii) $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (ax^2 - 3) = 9a - 3$

So for the limit to exist at -3, b must equal $9a - 3$. That is, $b = 9a - 3$.

(c) $\lim_{x \to 9} f(x) = f(9)$. By items (a) and (b), we have that this equality holds.

So our choice of constants are a a real number and $b = 9a - 3$. With these constants, $f(x)$ is continuous everywhere.