Homework 15

Name: **SOLUTIONS** Date: August 12, 2015

P 1. Consider the family of curves

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Find the orthogonal family of curves. For each family, sketch two trajectory curves.

Solution:

$$\frac{d}{dx}\left[\frac{x^2}{9} + \frac{y^2}{4}\right] = 0 \Rightarrow \frac{2x}{9} + \frac{2y}{4}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{4x}{9y}$$

So, the slope of the orthogonal family is given by

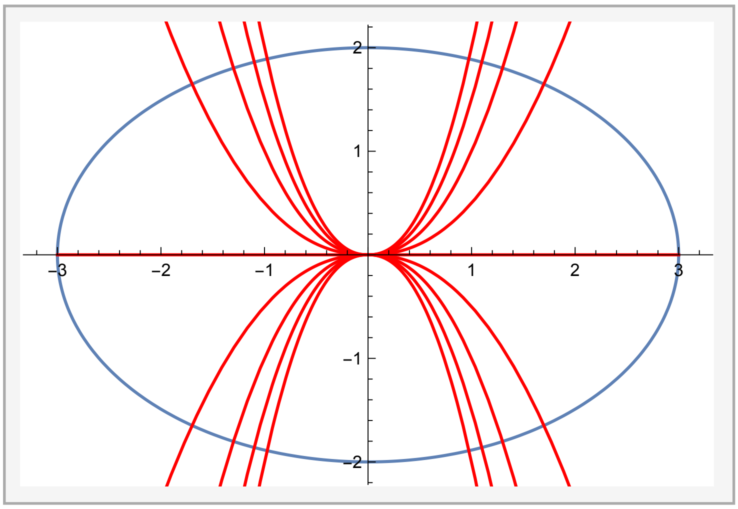
$$\frac{dy}{dx} = \frac{9y}{4x}$$

Solving this differetial equation, we get:

$$\frac{dy}{dx} = \frac{9y}{4x} \Rightarrow \int \frac{1}{9y}\,dy = \int \frac{1}{4x}\,dx \Rightarrow \frac{1}{9}\ln|y| = \frac{1}{4}\ln|x| + C_1$$

Solving explicitly for y, we obtain:

$$\ln y = \ln x^{9/4} + C_2 \Rightarrow y = Cx^{9/4}$$



P 2. A 100-gallon tank is full of a solution containing 25 pounds of a concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at a rate of 5 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

(a) Find the amount Q of the concentrate in the solution as a function of t.

(b) Find the time when the amount of concentrate in the tank reaches 15 pounds.

Solution: Let Q be the amount of concentrate in the tank at time t. Then dQ/dt is the rate of change of the amount of concentrate in the tank. This rate is negative since we are only losing losing concentrate over time.

$$\frac{dQ}{dt} = -\frac{Q}{100}\frac{\text{lbs}}{\text{gal}} \cdot 5\frac{\text{gal}}{\text{min}} = -\frac{Q}{20}$$

The solution to this differential equation is an exponential decay model.

$$Q = Ce^{-1/20t}.$$

But at time $t = 0$, there is 25 lbs. of concentrate and so $C = 25$.

(a) The amount Q of the concentrate in the solution as a function of time is

$$Q = 25e^{-1/20t}$$

(b) At $t = -20 \ln \frac{15}{25}$ the amount of concentrate in the tank reaches 15 lbs.

$$15 = 25e^{-1/20t} \Rightarrow t = -20 \ln \frac{15}{25} \approx 10.2165$$