Homework 10

Name: **SOLUTIONS** Date: August 5, 2015

P 1. Find the 5th Taylor polynomial of $f(x) = \frac{2}{x}$ centered at 1.

Solution:

$$
\begin{array}{lll}
f(x) = 2x^{-1} & \Rightarrow & f(1) = 2 \\
f'(x) = 2(-1)x^{-2} & \Rightarrow & f'(1) = 2(-1) = -2 \\
f''(x) = 2(-1)(-2)x^{-3} & \Rightarrow & f''(1) = 2(-1)(-2) \\
f'''(x) = 2(-1)(-2)(-3)x^{-4} & \Rightarrow & f'''(1) = 2(-1)(-2)(-3) \\
f^{(4)}(x) = 2(-1)(-2)(-3)(-4)x^{-5} & \Rightarrow & f^{(4)}(1) = 2(-1)(-2)(-3)(-4) \\
\vdots & \vdots & \vdots \\
f^{(n)}(x) = 2(-1)^n n! x^{-n-1} & \Rightarrow & f^{(n)}(1) = 2(-1)^n n!
\end{array}
$$

So,

$$
\begin{aligned}
P_5(x) &= \sum_{i=0}^{5} \frac{f^{(n)}(1)}{n!}(x-1)^n \\
&= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 \\
&= 2 - 2(x-1) + \frac{4}{2}(x-1)^2 + \frac{-12}{3!}(x-1)^3 + \frac{48}{4!}(x-1)^4 + \frac{-240}{5!}(x-1)^5 \\
&= 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3 + 2(x-1)^4 - 2(x-1)^5
\end{aligned}
$$

P 2. Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{\sqrt{n}}$$

Suppose $x \neq 3$. ***Solution***:

(a)

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{(n+1)+1}(x-3)^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^{n+1}(x-3)^n}{\sqrt{n}}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{(n+1)+1}(x-3)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^{n+1}(x-3)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{-1(x-3)^n(x-3)}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-3)^n}\right| \\
&= \lim_{n\to\infty}\left|(x-3)\cdot\sqrt{\frac{n}{n+1}}\right| \\
&= |x-3| \lim_{n\to\infty} \cdot \sqrt{\frac{n}{n+1}} \\
&= |x-3| < 1
\end{aligned}$$

By the Ratio Test, the series converges absolutely for

$$|x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$$

(b) If $x = 2$, then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{-1}{\sqrt{n}}$$

diverges by the p-series ($p = 1/2 \leq 1$) test.

(c) If $x = 4$, then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

converges by the Alternating Series Test.

The interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{\sqrt{n}}$$

is $(2, 4]$.