Homework 1

Name: **SOLUTIONS** Date: July 9, 2015

P 1. Find

$$\int (t^2 + t - 1)\sqrt{3t+4}\,dt$$

Solution: Let $u = 3t + 4$ so that $du = 3\,dt$ and $dt = \frac{du}{3}$. So,

$$\begin{aligned}
\int (t^2 + t - 1)\sqrt{3t+4}\,dt &= \int (t^2 + t - 1)\sqrt{u}\,\frac{du}{3} \\
&= \int \left(\left(\frac{u-4}{3} \right)^2 + \frac{u-4}{3} - 1 \right) \sqrt{u}\,\frac{du}{3} \\
&= \int \frac{u^{5/2}}{27} - \frac{5u^{3/2}}{27} - \frac{5\sqrt{u}}{27}\,du \\
&= \frac{2}{567} u^{3/2} \left(3u^2 - 21u - 35 \right) + C \\
&= \frac{2}{567} (3t+4)^{3/2} \left(3(3t+4)^2 - 21(3t+4) - 35 \right) + C
\end{aligned}$$

P 2. Find
$$\int 9^{2x+5} + 3x\sec(4x^2+2) - 5\cot\left(\frac{x}{11}\right)\,dx$$

Solution:
$$\int 9^{2x+5}\,dx = \int 9^u\,\frac{du}{2} = \frac{1}{2}\frac{9^u}{\ln 9} + C_1 = \frac{9^{2x+5}}{2\ln 9} + C_1 \tag{1}$$
$$\begin{aligned}\int 3x\sec(4x^2+2)\,dx &= \int \frac{3}{8}\sec u\,du \\ &= \frac{3}{8}\ln|\sec u + \tan u| + C_2 \\ &= \frac{3}{8}\ln|\sec(4x^2+2)+\tan(4x^2+2)| + C_2\end{aligned} \tag{2}$$
$$\int 5\cot\left(\frac{x}{11}\right)\,dx = 5\int \cot u\;11du = 55\ln|\sin u| + C_3 = 55\ln|\sin(x/11)| + C_3 \tag{3}$$
So, by (**??**), (**??**), and (**??**) we have,
$$\begin{aligned}\int 9^{2x+5} + 3x\sec(4x^2+2) - 5\cot\left(\frac{x}{11}\right)\,dx &= \int 9^{2x+5}\,dx + \int 3x\sec(4x^2+2)\,dx - \int 5\cot\left(\frac{x}{11}\right)\,dx \\ &= \frac{9^{2x+5}}{2\ln 9} + \frac{3}{8}\ln|\sec(4x^2+2)+\tan(4x^2+2)| \\ &\quad - 55\ln|\sin(x/11)| + C\end{aligned}$$

[Note: $C = C_1 + C_2 - C_3$]

P 3. Use the Trapezoidal Rule to approximate

$$\int_0^3 \frac{1}{60}\left(\frac{x^6}{2} - 4x^5 + \frac{15x^4}{2}\right)\, dx,$$

with 5 trapezoids.

(a) What is a bound on the error in using this approximation?

(b) What is the exact value?

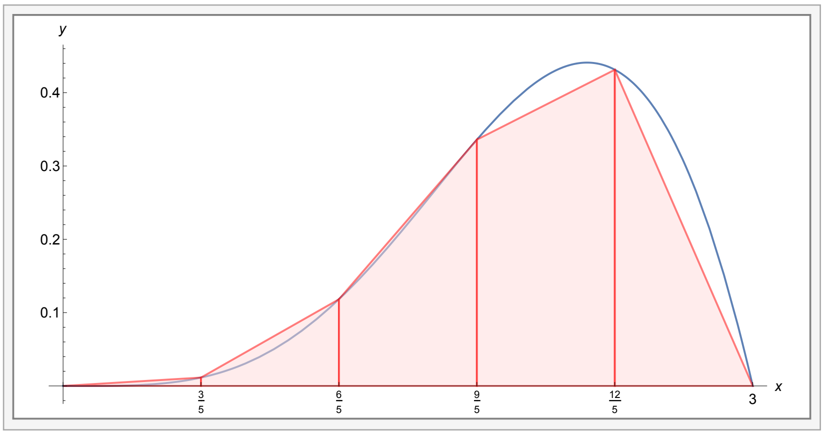
(c) What is the exact error?

Solution: The approximate value of the definite integral using the Trapezoidal Rule with $n = 5$ is

$$T_5 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_4) + f(x_5)\right]$$

where $x_i = a + i\Delta x$. So,

$$T_5 = \frac{42039}{78125} \approx 0.538099$$



A bound on the error using the Trapezoidal Rule is given by

$$|E_n| \le \frac{M(b-a)^3}{12n^2},$$

where M is a bound on $|f''(x)|$ on $[0, 3]$. So, if we find the extreme values of $g(x) = f''(x)$ on $[0, 3]$ then the largest of these in absolute value is the best possible choice for M.

$$f'(x) = \frac{1}{60}\left(3x^5 - 20x^4 + 30x^3\right)$$
$$g(x) = f''(x) = \frac{1}{12}\left(3x^4 - 16x^3 + 18x^2\right)$$

Then

$$g'(x) = x^3 - 4x^2 + 3x = (x-3)(x-1)x,$$

The solutions for $g'(x) = 0$ are $x = 0, 1$, and $x = 3$. We compare the values of $|g(x)|$ at $x = 0, 1, 3$, and the endpoints of the interval. M will be the largest of these values.

x	$\|g(x)\|$
0	$0 \approx 0.$
1	$\frac{5}{12} \approx 0.416667$
3	$\frac{9}{4} \approx 2.25$

So $M = 9/4$ Thus,

$$|E_5| \leq \frac{9/4(3)^3}{12 \cdot 5^2} \leq 0.2025$$

The exact value of the integral is

$$\int_0^3 \frac{1}{60}\left(\frac{x^6}{2} - 4x^5 + \frac{15x^4}{2}\right)\, dx = \frac{81}{140} \approx 0.578571.$$

The actual error in approximating using the Trapezoidal Rule is

$$|\text{Actual} - \text{Approximation}| \approx |0.578571 - 0.538099| = 0.0404722$$

which is as expected (i.e. within the error bound).