Exam 3

Name: **SOLUTIONS** Date: August 11, 2015

P 1 (2 Points)**.** State the Divergence Test.

Solution: If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ is divergent.

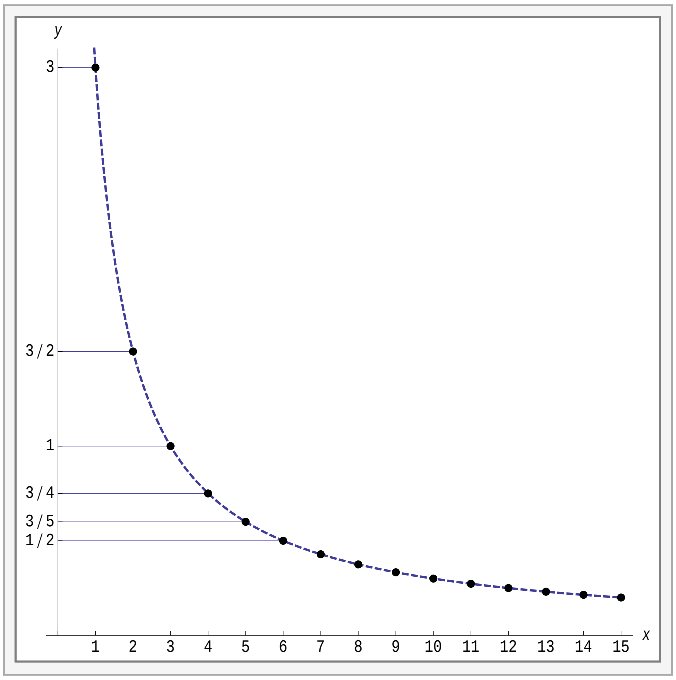
P 2 (2 Points)**.** State the Integral Test.

Solution: Let $f(x)$ be a positive, continuous, decreasing function on $[1, \infty)$. Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge.

P 3. [4 Points] Consider the graph of f below.



Use the graph of f to answer the following.

(a) Consider the sequence

$$a_n = f(n),\ n \geq 1.$$

Find $\lim_{n\to\infty} a_n$.

Solution:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{n} = 0$$

(b) Consider the sequence

$$b_n = \cos[f(n)],\ n \geq 1.$$

Find $\lim_{n\to\infty} b_n$.

Solution:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cos\frac{3}{n} = 1$$

(c) Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} f(n)$$

Solution:

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{3}{n}$$

diverges as a harmonic series.

(d) Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} [f(n)]^2$$

Solution:

$$\sum_{n=1}^{\infty} [f(n)]^2 = \sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^2 = \sum_{n=1}^{\infty} \frac{9}{n^2}$$

converges by the p-series test ($p = 2$).

P 4 (10 Points)**.** Determine whether the series converges or diverges. If it converges, find its sum.

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^n}$$

Solution: Converges by the geometric series test with $a = 3$ and

$$r = \frac{a_{n+1}}{a_n} = \frac{3^{n+2}}{7^{n+1}} \frac{7^n}{3^{n+1}} = \frac{3}{7} < 1.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^n} = \frac{a}{1-r} = \frac{3}{1-3/7} = 21/4.$$

P 5 (10 Points)**.** Determine whether the series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$\begin{aligned}
s_1 &= 1 - \frac{1}{2} \\
s_2 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \\
s_3 &= 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \\
s_4 &= 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = 1 - \frac{1}{5} \\
&\vdots \\
s_n &= 1 - \frac{1}{n+1}
\end{aligned}$$

So,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

Therefore, the series converges and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

P 6 (10 Points)**.** Determine whether the series converges or diverges, explain why.

$$\sum_{n=0}^{\infty} \frac{2n+1}{3n+2}$$

Solution: Note,

$$\lim_{n\to\infty} \frac{2n+1}{3n+2} \overset{\infty/\infty}{=} 2/3$$

So, by the Divergence Test

$$\sum_{n=0}^{\infty} \frac{2n+1}{3n+2}$$

diverges.

P 7 (10 Points)**.** Determine whether the series converges or diverges, explain why.

$$\sum_{n=1}^{\infty} ne^{-n}$$

Solution: Ratio Test:

- $a_n \neq 0$
- $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{ne}\right| = \frac{1}{e} < 1.$$

By the ratio test,

$$\sum_{n=1}^{\infty} ne^{-n}$$

converges absolutely and so converges.

P 8 (10 Points)**.** Determine whether the series converges absolutely, conditionally, or diverges, and explain why.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Solution: Alternating Series Test:

- Let $a_n = 1/\sqrt{n} > 0$.
- $\lim_{n\to\infty} a_n = 0$.
- $a_{n+1} \leq a_n$:

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \Leftrightarrow \sqrt{n} \leq \sqrt{n+1} \Leftrightarrow n \leq n+1 \Leftrightarrow 0 \leq 1.$$

So, by the alternating series test,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges. Absolute Convergence Test:

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges by the p-series test ($p = 1/2 < 1$). Therefore,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges conditionally.

P 9 (10 Points)**.** Find the third Maclaurin polynomial of $f(x) = e^{-3x}$.

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} \Rightarrow e^{-3x} \approx \sum_{n=0}^{3} \frac{(-3x)^n}{n!}$$

So, the third Maclaurin polynomial of $f(x) = e^{-3x}$ is

$$\sum_{n=0}^{3} \frac{(-3x)^n}{n!} = 1 - 3x + \frac{9}{2!}x^2 - \frac{27}{3!}x^3 = 1 - 3x + \frac{9}{2!}x^2 - \frac{9}{2}x^3$$

P 10 (10 Points)**.** Find Taylor series of

$$f(x) = \frac{1}{x}$$

centered at 1.

Solution:

$$f(x) = \frac{1}{x} = \frac{1}{1+(x-1)} = \frac{1}{1-[-(x-1)]} = \sum_{n=0}^{\infty}[-(x-1)]^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

So, the Taylor series of $f(x)$ centered at 1 is

$$\sum_{n=0}^{\infty}(-1)^n(x-1)^n.$$

P 11 (10 Points)**.** Determine if the series converges or diverges, explain why.

$$\sum_{n=1}^{\infty}\frac{2}{6n+1}$$

Solution: Limit Comparison Test

- Let $a_n = \frac{2}{6n+1}$.
- Comparison Series:
$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{2}{6n} = \sum_{n=1}^{\infty}\frac{1}{3n}$$
- $$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{2}{6n+1}}{\frac{1}{3n}} = \lim_{n\to\infty}\frac{6n}{6n+1} = 1$$

So, by the Limit Comparison Test,

$$\sum_{n=1}^{\infty}\frac{2}{6n+1}$$

diverges.

P 12 (12 Points)**.** Determine the interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}}$$

Solution: Ratio Test

- $a_n \neq 0$ for $x \neq 3$.
- $$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(x-3)\sqrt{n}}{\sqrt{n+1}}\right| \\ &= |x-3| \lim_{n\to\infty}\left|\sqrt{\frac{n}{n+1}}\right| \\ &= |x-3| < 1\end{aligned}$$
- $|x-3| < 1 \Leftrightarrow -1 < x-3 < 1 \Leftrightarrow 2 < x < 4 \Leftrightarrow (2,4)$
- For $x = 2$,
  $$\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(2-3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$
  which converges by the alternating series test.
- For $x = 4$,
  $$\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(4-3)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$
  which diverges by the p-series test ($p = 1/2 < 1$).

So, the interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n}}$$

is $[2,4)$.

P 13 (Bonus 2 Points)**.** Find the limit

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

Solution: Note,

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

So,

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1}{x}&=\lim_{x\to 0}\frac{\sum_{n=0}^{\infty}\frac{x^n}{n!}-1}{x}=\lim_{x\to 0}\frac{1+x+\frac{x^2}{2}+\cdots-1}{x}=\lim_{x\to 0}\frac{x+\frac{x^2}{2}+\cdots}{x}\\&=\lim_{x\to 0}\left(1+\frac{x}{2}+\cdots\right)=1+0+0+\cdots=1\end{aligned}$$

P 14 (Bonus 2 Points)**.** Find the sum

$$\sum_{n=0}^{\infty}\frac{1}{5^n n!}$$

Solution: Note,

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

So,

$$\sum_{n=0}^{\infty}\frac{1}{5^n n!}=\sum_{n=0}^{\infty}\frac{(1/5)^n}{n!}=e^{1/5}$$