Exam 2

Name: **SOLUTIONS** Date: July 27, 2015

P 1 (2 Points)**.** State L'Hospital's Rule.

Solution: Suppose that f and g are differentiable in an open interval I that contains a, with the possible exception of a itself, and $g'(x) \neq 0$ for all x in I. If $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of the type $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists or is infinite.

P 2 (2 Points)**.** Circle all items in the list that have indeterminate form.

(a) $\boxed{1^\infty}$

(b) $\infty \cdot \infty$

(c) $\boxed{\infty - \infty}$

(d) $\infty + \infty$

(e) $\boxed{\infty - 3\infty}$

(f) $\boxed{0 \cdot \infty}$

(g) $\boxed{0/0}$

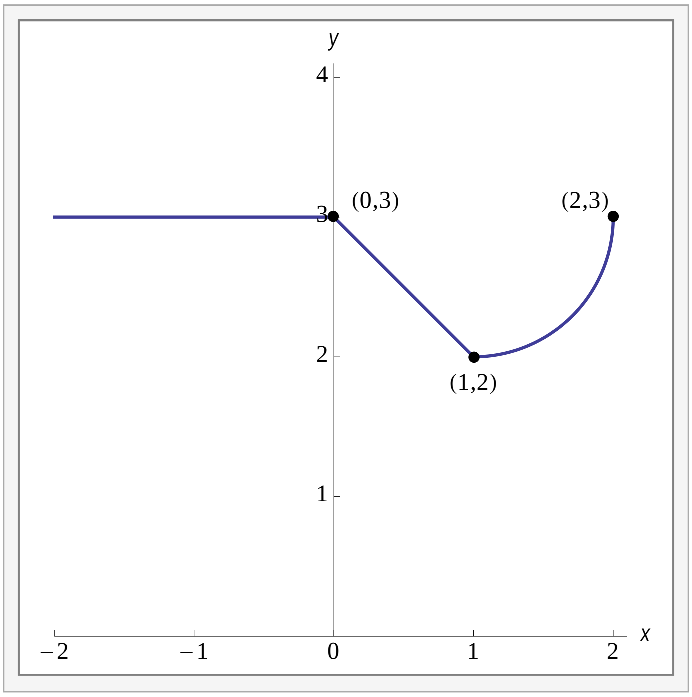
(h) $3/0$

(i) $\boxed{\dfrac{\infty}{-\infty}}$

(j) $\frac{500000}{\infty}$

(k) $\boxed{0^0}$

P 3. [4 Points] Consider the graph of f below.



Use the graph of f to answer the following. If a solution does not exist, state why.

(a) Find the area of the region between the graphs of

$$y = f(x) \text{ and } y = x + 1,$$

on $[0, 1]$.

Solution:

$$A = \frac{1}{2} b \cdot h = \frac{1}{2}(2)(1) = 1$$

(b) Find the arc length of the graph of $y = f(x)$ on $[-2, 0]$.

Solution:

$$s = 0 - (-2) = 2$$

(c) Find the volume of the solid of revolution obtained by revolving the region between the graphs of $y = f(x)$ and $y = 3$, on $[1, 2]$ about the line $y = 3$.

Solution:

$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi$$

(d) Find the surface area of the surface of revolution obtained by revolving the part of the graph of $y = f(x)$ from $x = 0$ to $x = 1$, about the y-axis.

Solution:

$$S = 2\pi r \cdot l = 2\pi \frac{1}{2}\sqrt{2} = \sqrt{2}\pi$$

P 4 (10 Points)**.** Evaluate

$$\int_0^\infty xe^{-x}\,dx$$

Solution:

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C = -e^{-x}(1+x) + C = \frac{-(1+x)}{e^x} + C$$

$$\int_0^b xe^{-x}\,dx = \frac{-(1+x)}{e^x}\Big|_0^b = \frac{-(1+b)}{e^b} + 1$$

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty}\left(\frac{-(1+b)}{e^b} + 1\right) = 1$$

P 5 (10 Points)**.** Evaluate

$$\lim_{x\to 0^+} x^x$$

Solution:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = e^{\lim_{x\to 0^+} x\ln x}$$

But,

$$\lim_{x\to 0^+} x\ln x \overset{0\cdot(-\infty)}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{-\infty/\infty}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

So,

$$\lim_{x\to 0^+} x^x = e^0 = 1.$$

P 6 (10 Points). Find

$$\int \frac{2x^2 - x + 7}{(x+1)(x^2+4)}\, dx$$

Solution:

$$\frac{2x^2 - x + 7}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

x	$2x^2 - x + 7 = A(x^2+4) + (Bx+C)(x+1)$
-1	$10 = 5A \Rightarrow A = 2$
0	$7 = 4A + C \Rightarrow 7 = 8 + C \Rightarrow C = -1$
1	$8 = 5A + (B+C)(2) \Rightarrow 8 = 10 + 2(B-1) \Rightarrow B = 0$

So,

$$\begin{aligned}
\int \frac{2x^2 - x + 7}{(x+1)(x^2+4)}\, dx &= \int \frac{2}{x+1} - \frac{1}{x^2+4}\, dx \\
&= 2\ln|x+1| - \int \frac{1}{x^2+4}\, dx \\
&= 2\ln|x+1| - \frac{1}{4}\int \frac{1}{(x/2)^2+1}\, dx \\
&= 2\ln|x+1| - \frac{1}{2}\int \frac{1}{u^2+1}\, du \\
&= 2\ln|x+1| - \frac{1}{2}\arctan u + C \\
&= 2\ln|x+1| - \frac{1}{2}\arctan(x/2) + C
\end{aligned}$$

P 7 (10 Points). Find

$$\int \frac{5}{\sqrt{1-4x^2}}\, dx$$

Solution:

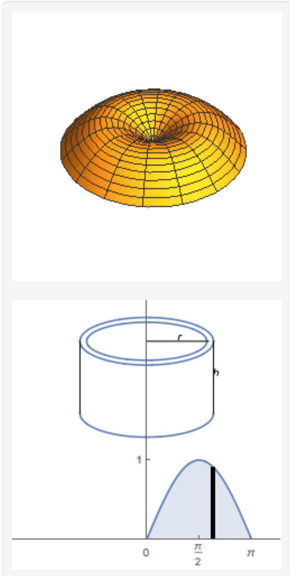
$$\begin{aligned}
\int \frac{5}{\sqrt{1-4x^2}}\, dx &= \int \frac{5}{\sqrt{1-4(1/2\sin t)^2}}\, \frac{1}{2}\cos t\, dt \\
&= \frac{5}{2}\int \frac{1}{\sqrt{1-\sin^2 t}}\, \cos t\, dt \\
&= \frac{5}{2}\int \frac{1}{\sqrt{\cos^2 t}}\, \cos t\, dt \\
&= \frac{5}{2}\int \frac{1}{|\cos t|}\, \cos t\, dt \\
&= \frac{5}{2}\int \frac{1}{\cos t}\, \cos t\, dt = \frac{5}{2}\int dt \\
&= \frac{5}{2}t + C = \frac{5}{2}\arcsin(2x) + C
\end{aligned}$$

P 8 (10 Points). Find the volume of the solid of revolution obtained by revolving the region between by the graphs of

$$y = \sin x \text{ and } y = 0$$

on the interval $[0, \pi]$, about the y-axis.

Solution:



$$\Delta V = 2\pi r h \Delta x = 2\pi x \sin x \Delta x \text{ for } 0 \le x \le \pi$$

Note,

$$\int x \sin x \, dx = -x\cos x + \sin x + C.$$

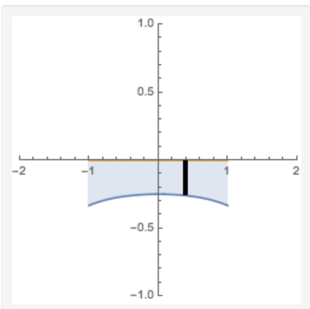
So,

$$\begin{aligned} V &= \int_0^{\pi} 2\pi x \sin x \, dx \\ &= 2\pi(-x\cos x + \sin x)\Big|_0^{\pi} \\ &= 2\pi(-\pi\cos\pi + \sin\pi) \\ &= 2\pi^2 \end{aligned}$$

P 9 (10 Points). Find the area of the region bounded by

$$y = \frac{1}{x^2 - 4}, \ y = 0, \ x = -1, \quad \text{and } x = 1.$$

Solution:



$$\Delta A = h \cdot b = \left(0 - \frac{1}{x^2 - 4}\right) \cdot \Delta x \text{ for } -1 \le x \le 1$$

$$\begin{aligned} \int \frac{1}{x^2 - 4} &= \int \frac{1}{(x-1)(x+1)} \, dx \\ &= \int \frac{1}{4(x-2)} - \frac{1}{4(x+2)} \, dx \\ &= \frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right) + C \\ &= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C \end{aligned}$$

$$\begin{aligned} A &= \int_{-1}^{1} 0 - \frac{1}{x^2 - 4} \, dx \\ &= -2\int_0^1 \frac{1}{x^2 - 4} \, dx \\ &= -\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|\Big|_0^1 \\ &= -\frac{1}{2}\ln(1/3) = \frac{1}{2}\ln 3 \end{aligned}$$

P 10 (10 Points)**.** Evaluate

$$\lim_{x\to 2^+}\left(\frac{8}{x^2-4}-\frac{2}{x-2}\right)$$

Solution:

$$\begin{aligned}\lim_{x\to 2^+}\left(\frac{8}{x^2-4}-\frac{2}{x-2}\right) &\overset{\infty-\infty}{=} \lim_{x\to 2^+}\left(\frac{8}{x^2-4}-\frac{2(x+2)}{x^2-4}\right)\\ &= \lim_{x\to 2^+}\frac{-2x+4}{x^2-4}\\ &\overset{0/0}{=} \lim_{x\to 2^+}\frac{-2}{2x}\\ &= \frac{-1}{2}\end{aligned}$$

P 11 (10 Points)**.** Find

$$\int \sin^2 x\cos^2 x\ dx$$

Solution:

$$\begin{aligned}\int \sin^2 x\cos^2 x\ dx &= \int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)\ dx\\ &= \frac{1}{4}\int 1-\cos^2 2x\ dx\\ &= \frac{1}{4}\left(x-\int \cos^2 2x\ dx\right)\\ &= \frac{1}{4}\left(x-\int \frac{1+\cos 4x}{2}\ dx\right)\\ &= \frac{1}{4}x-\frac{1}{8}\int 1+\cos 4x\ dx\\ &= \frac{1}{4}x-\frac{1}{8}\left(x+\frac{\sin 4x}{4}\right)+C\\ &= \frac{1}{8}x-\frac{\sin 4x}{32}+C\end{aligned}$$

P 12 (6 Points)**.** Determine if the improper integral converges or diverges. If it converges, find its value.

$$\int_0^8 \frac{3}{\sqrt{8-x}}\, dx$$

Solution: Note,

$$\begin{aligned}\int \frac{3}{\sqrt{8-x}}\, dx &= \int \frac{3}{\sqrt{u}}\, \frac{du}{-1} \\ &= -3 \int u^{-1/2}\, du \\ &= -6\sqrt{u} + C \\ &= -6\sqrt{8-x} + C\end{aligned}$$

and

$$\begin{aligned}\int_0^b \frac{3}{\sqrt{8-x}}\, dx &= -6\sqrt{8-x}\Big|_0^b \\ &= -6\sqrt{8-b} + 12\sqrt{2}\end{aligned}$$

So,

$$\begin{aligned}\int_0^8 \frac{3}{\sqrt{8-x}}\, dx &= \lim_{b\to 8^-} \int_0^b \frac{3}{\sqrt{8-x}}\, dx \\ &= \lim_{b\to 8^-} \left(-6\sqrt{8-b} + 12\sqrt{2}\right) \\ &= 12\sqrt{2} - 6 \lim_{b\to 8^-} \sqrt{8-b} \\ &= 12\sqrt{2}\end{aligned}$$

P 13 (6 Points)**.** Find

$$\int_0^\pi x\cos x\, dx$$

Solution:

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx = x\sin x + \cos x + C$$

So,

$$\int_0^\pi x\cos x\, dx = (x\sin x + \cos x)\Big|_0^\pi = \pi\sin\pi + \cos\pi - (0\sin 0 + \cos 0) = -1 - 1 = -2$$