Exam 1

Name: **SOLUTIONS** Date: July 22, 2015

P 1. [2 Points] State the Trapezoidal Rule and its assumptions.

Solution: Let f be continuous on $[a, b]$. Then

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)],$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.

P 2. [2 Points] State the Integration by Parts formula and its assumptions.

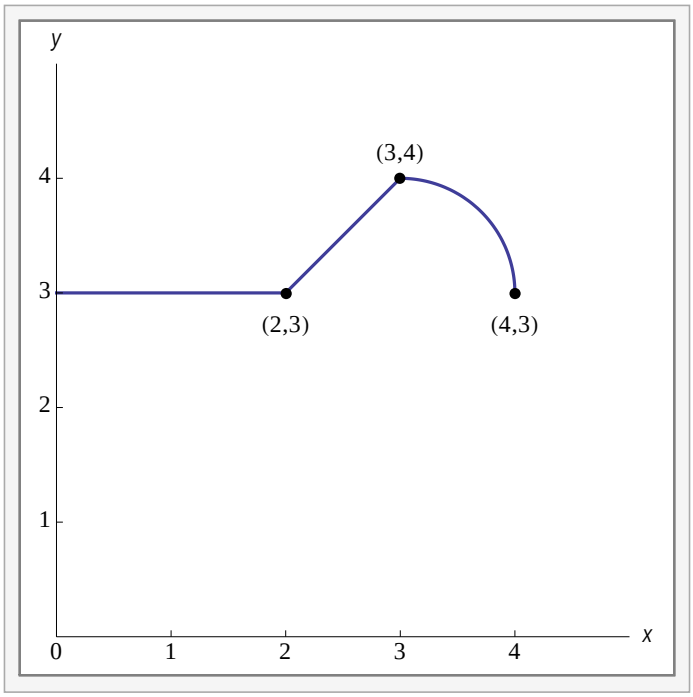
Solution: If $f(x)$ and $g(x)$ have continuous derivatives, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

or, equivalently,

$$\int u\,dv = uv - \int v\,du.$$

P 3. [4 Points] Consider the graph of f below.



Use the graph of f to answer the following. If a solution does not exist, state why.

(a) Find the area of the region bounded by

$$y = f(x), y = 1, x = 1, x = 3$$

Solution: $9/2$

(b) Find the length of the graph of $y = f(x)$ on $[2, 4]$.

Solution: $\sqrt{2} + \frac{1}{4}(2\pi(1)) = \sqrt{2} + \frac{\pi}{2}$

(c) Find the volume of the solid of revolution obtained by revolving the region bounded by the graph of $y = f(x)$, $y = 0$, $x = 0$, and $x = 2$, about the x-axis.

Solution: $V = \pi r^2 h = \pi(3)^2 \cdot 2 = 18\pi$

(d) Find the surface area of the surface of revolution obtained by revolving the part of the graph of $y = f(x)$ from $x = 2$ to $x = 3$, about the x-axis.

Solution: $S = 2\pi rl = 2\pi(3.5)\sqrt{2} = 7\sqrt{2}\pi$

P 4 (10 Points)**.** Find

$$\int x^5 \ln 3x \, dx$$

Solution: Let $u = \ln 3x$ and $dv = x^5$. So, $du = \frac{1}{x}$ and $v = \frac{x^6}{6}$. By Integration-by-Parts,

$$\begin{aligned}\int x^5 \ln 3x \, dx &= uv - \int v \, du \\ &= \frac{x^6 \ln 3x}{6} - \int \frac{x^5}{6} \, dx \\ &= \frac{x^6 \ln 3x}{6} - \frac{x^6}{36} + C\end{aligned}$$

P 5 (10 Points)**.** Find

$$\int x^3 e^x \, dx$$

Solution:

$$\begin{array}{cc} x^3 & e^x \\ \overset{+1}{\searrow} & \\ 3x^2 & e^x \\ \overset{-1}{\searrow} & \\ 6x & e^x \\ \overset{+1}{\searrow} & \\ 6 & e^x \\ \overset{-1}{\searrow} & \\ 0 & e^x \end{array}$$

So,

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

P 6 (10 Points). Find

$$\int e^x \sin x \, dx$$

Solution:

$$\begin{array}{lll} e^x & \overset{+1}{\searrow} & \sin x \\ e^x & \overset{-1}{\searrow} & -\cos x \\ e^x & \overset{+1}{\rightarrow} & -\sin x \end{array}$$

So,

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx.$$

That is,

$$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x.$$

Thus,

$$\int e^x \sin x \, dx = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

P 7 (10 Points). Find

$$\int \sin(5x)\cos(2x) \, dx$$

[Hint: $\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$]

Solution:

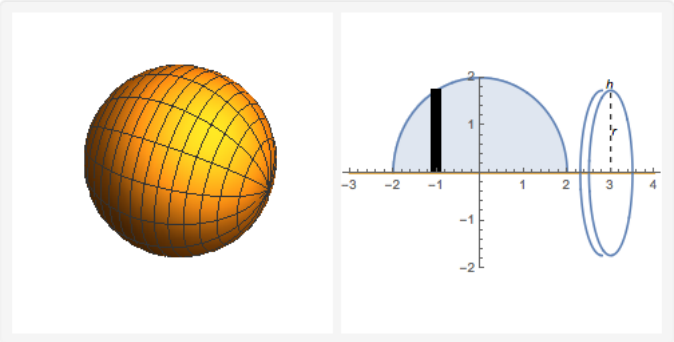
$$\begin{aligned} \int \sin(5x)\cos(2x) \, dx &= \int \frac{1}{2}(\sin[(5-2)x] + \sin[(5+2)x]) \, dx \\ &= \frac{1}{2}\int \sin[3x] + \sin[7x] \, dx \\ &= \frac{1}{2}\left(\frac{-1}{3}\cos[3x] - \frac{1}{7}\cos[7x]\right) + C \\ &= -\frac{1}{2}\left(\frac{1}{3}\cos[3x] + \frac{1}{7}\cos[7x]\right) + C \end{aligned}$$

P 8 (6 Points). Find the volume of the solid obtained by revolving the region bounded by

$$y = \sqrt{4 - x^2} \text{ and } y = 0$$

about the x-axis.

Solution:

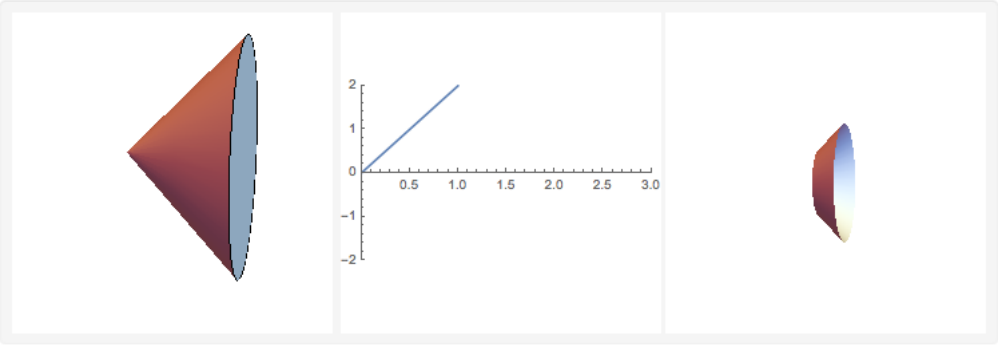


$$\begin{aligned}\Delta V &= \pi r^2 h \\ &= \pi(\sqrt{4-x^2})^2\ \Delta x \\ &= \pi|4-x^2|\ \Delta x \\ &= \pi(4-x^2)\ \Delta x, \text{ for } -2 \le x \le 2\end{aligned}$$

$$\begin{aligned}V &= \int_{-2}^{2} \pi(4-x^2)\ dx = 2\int_{0}^{2} \pi(4-x^2)\ dx \\ &= 2\pi\left(4x - \frac{x^3}{3}\right)\Big|_0^2 \\ &= 2\pi\left(4(2) - \frac{2^3}{3}\right) = 2\pi\frac{16}{3} = \frac{32\pi}{3}\end{aligned}$$

P 9 (6 Points). Find the surface area of the surface of revolution obtained by revolving the graph of $y = 2x$ defined on $[0, 1]$, about the x-axis.

Solution:



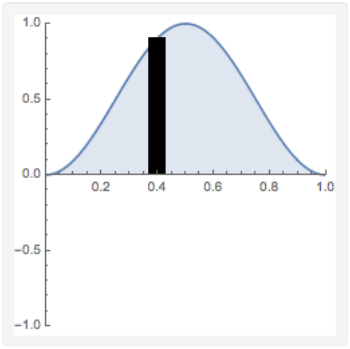
$$\begin{aligned}\Delta S &= 2\pi r l \\ &= \pi f(x)\sqrt{1+[f'(x)]^2}\ \Delta x \\ &= \pi 2x\sqrt{1+2^2}\ \Delta x \\ &= 2\sqrt{5}\pi\ \Delta x, \text{ for } 0 \le x \le 1\end{aligned}$$

$$\begin{aligned}V &= \int_0^1 2\sqrt{5}\pi\ dx \quad = 2\sqrt{5}\pi x\Big|_0^1 \\ &= 2\sqrt{5}\pi\end{aligned}$$

P 10 (10 Points)**.** Find the area of the region bounded by the graphs of the equations

$$y = \sin^2 \pi x,\ y = 0,\ x = 0,\ x = 1.$$

Solution:



$$\Delta A = h \cdot b$$
$$= \sin^2(\pi x)\ \Delta x,\ \text{for } 0 \le x \le 1$$

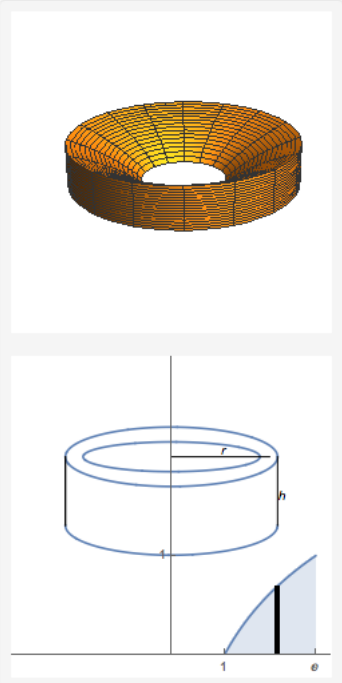
$$A = \int_0^1 \sin^2(\pi x)\ dx = \int_0^1 \frac{1 - \cos 2\pi x}{2}\ dx$$
$$= \frac{1}{2}\left(x - \frac{\sin 2\pi x}{2\pi}\right)\Big|_0^1$$
$$= 1/2$$

P 11 (10 Points)**.** Find the volume of the solid obtained by revolving the region bounded by the graphs of

$$y = \ln x,\ y = 0,\ \text{and } x = e$$

about the y-axis.

Solution:



$$\Delta V = 2\pi r h \cdot \Delta x$$
$$= 2\pi x \ln x,\ \text{for } 1 \le x \le e$$

$$V = \int_1^e 2\pi x \ln x\ dx = 2\pi \int_1^e x \ln x\ dx$$
$$= 2\pi \left(\frac{x^2}{2}\ln x\Big|_1^e - \int_1^e \frac{x}{2}\ dx\right)$$
$$= 2\pi \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_1^e$$
$$= 2\pi \left(\frac{e^2}{2} - \frac{e^2}{4}\right) - 2\pi(-1/4)$$
$$= 2\pi(e^2/4) + 2\pi/4 = \frac{\pi}{2}(e^2 + 1)$$

P 12 (10 Points). Find

$$\int_0^1 \arctan x \, dx$$

Solution:

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \ln |1+x^2| + C$$

So,

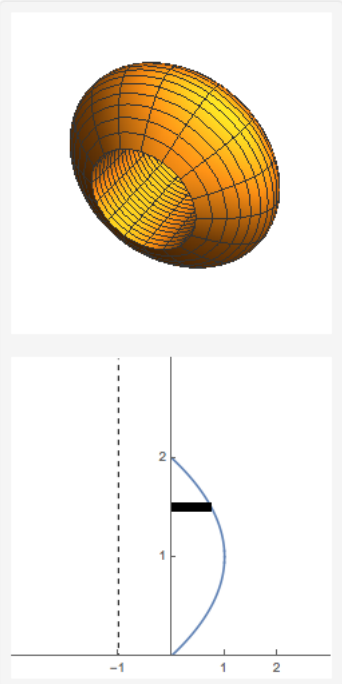
$$\int_0^1 \arctan x \, dx = \left(x \arctan x - \frac{1}{2} \ln |1+x^2| \right) \Big|_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2$$

P 13 (10 Points). Find the volume of the solid obtained by revolving the region bounded by the graphs of

$$x = -y^2 + 2y \text{ and } x = 0$$

about the line $x = -1$.

Solution:



$$\begin{aligned} \Delta V &= \pi(r_o^2 - r_i^2)\, \Delta y \\ &= \pi((-y^2+2y+1)^2 - 1^2), \text{ for } 0 \le y \le 2 \end{aligned}$$

$$\begin{aligned} V &= \int_0^2 \pi((-y^2+2y+1)^2 - 1)\, dy \\ &= \int_0^2 \pi(y^4 - 4y^3 + 2y^2 + 4y)\, dy \\ &= \pi \left(\frac{y^5}{5} - y^4 + \frac{2y^3}{3} + 2y^2 \right) \Big|_0^2 \\ &= \pi \left(\frac{2^5}{5} - 2^4 + \frac{2 \cdot 2^3}{3} + 2^3 \right) \\ &= \frac{56}{15}\pi \end{aligned}$$