2.2 Separable Equations

Name: Date: May 28, 2013

P 7. Solve

$$\frac{dx}{dt} = 3xt^2$$

P 8. Solve

$$x\frac{dy}{dx} = \frac{1}{y^3}$$

P 13. Solve

$$\frac{dx}{dt} - x^3 = x$$

P 15. Solve

$$y^{-1}\,dy + ye^{\cos x}\sin x\,dx = 0$$

P 16. Solve

$$(x + xy^2)\,dx + e^{x^2}y\,dy = 0$$

P 18. Solve

$$\frac{dy}{dx} = (1 + y^2)\tan x, \quad y(0) = \sqrt{3}$$

P 20. Solve

$$x^2 \frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)}, \quad (1) = 1$$

P 26. Solve

$$\sqrt{y}\, dx + (1+x)\, dy = 0, \quad y(0) = 1$$

P 27. Solutions Not Expressible in Terms of Elementary Functions. As discussed in calculus, certain indefinite integrals (antiderivatives) such as $\int e^{x^2}\,dx$ cannot be expressed in finite terms using elementary functions. When such an integral is encountered while solving a differential equation, it is often helpful to use definite integration (integrals with variable upper limit). For example, consider the initial value problem

$$\frac{dy}{dx} = e^{x^2} y^2, \quad y(2) = 1.$$

The differential equation separates if we divide by y^2 and multiply by dx. We integrate the separated equation from $x = 2$ to $x = x_1$ and find

$$\begin{aligned} \int_{x=2}^{x=x_1} e^{x^2}\,dx &= \int_{x=2}^{x=x_1} \frac{dy}{y^2} \\ &= -\frac{1}{y}\bigg|_{x=2}^{x=x_1} \\ &= -\frac{1}{y(x_1)} + \frac{1}{y(2)}. \end{aligned}$$

If we let t be the variable of integration and replace x_1 by x and $y(2)$ by 1, then we express the solution to the initial value problem by

$$y(x) = \left(1 - \int_2^x e^{t^2}\,dt\right)^{-1}.$$

Use definite integration to find an explicit solution to the initial value problems in parts (a)-(c).

(a) $\dfrac{dy}{dx} = e^{x^2}, \quad y(0) = 0$

(b) $\dfrac{dy}{dx} = e^{x^2} y^{-2}, \quad y(0) = 1$

(c) $\dfrac{dy}{dx} = \sqrt{1 + \sin x}\,(1 + y^2), \quad y(0) = 1$

P 34. Newton's Law of Cooling. According to Newton's law of cooling, if an object at temperature T is immersed in a medium having the constant temperature M, then the rate of change of T is proportional to the difference of temperature $M - T$. This gives the differential equation

$$\frac{dT}{dt} = k(M - T).$$

(a) Solve the differential equation for T.

(b) A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 min, the thermometer reads 80°F. What is the reading after 20 min?

P 38. Free Fall. A model for an object falling toward Earth is described by

$$m\frac{dv}{dt} = mg - bv,$$

assuming that only air resistance and gravity are acting on the object and v, m, g, and $b > 0$ are the velocity, mass, gravity, and some constant, respectively. If $m = 100$kg, $g = 9.8$ m/sec^2, $b = 5$ kg/sec, and $v(0) = 10$ m/sec, solve for $v(t)$. What is the limiting (i.e. terminal) velocity of the object?